Buckling of Confined Thin Sheets: from rigid walls to liquid walls

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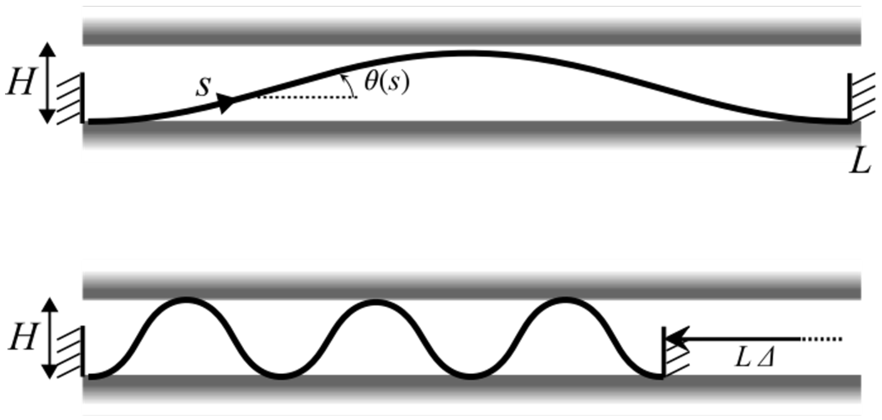

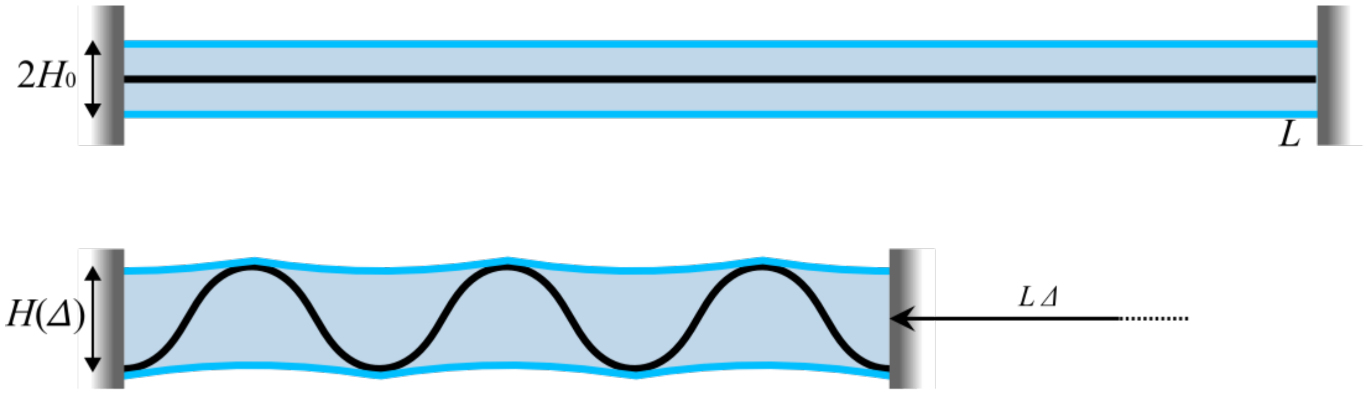

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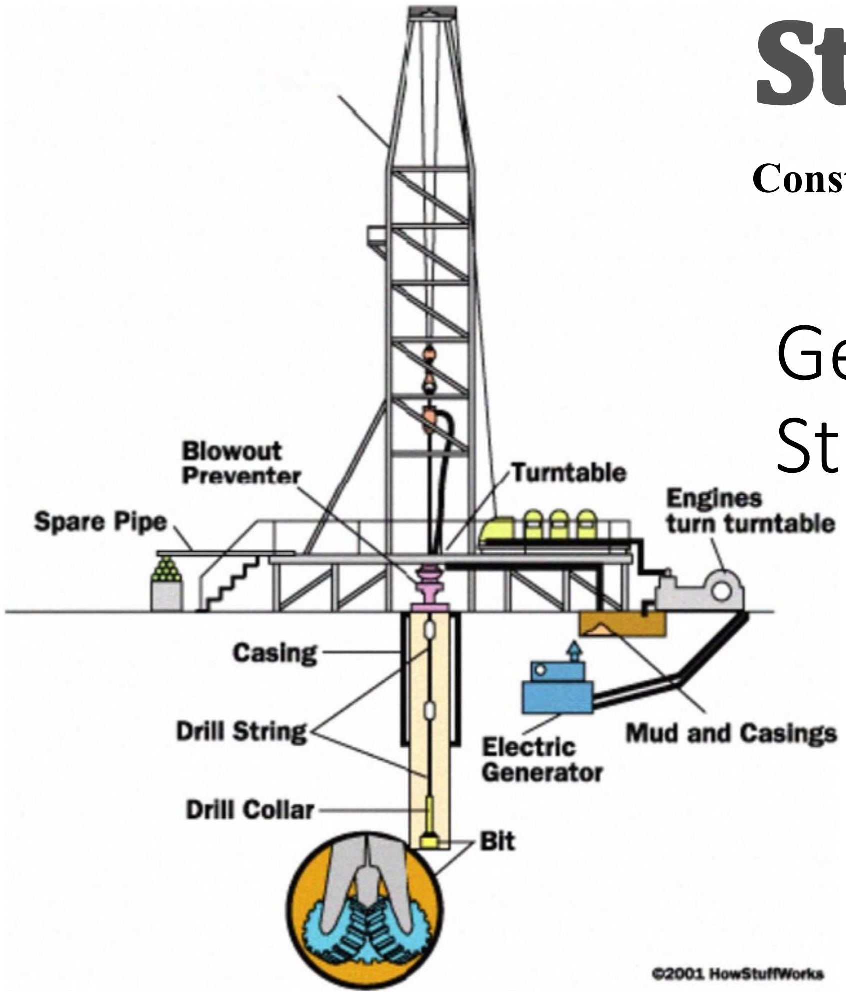
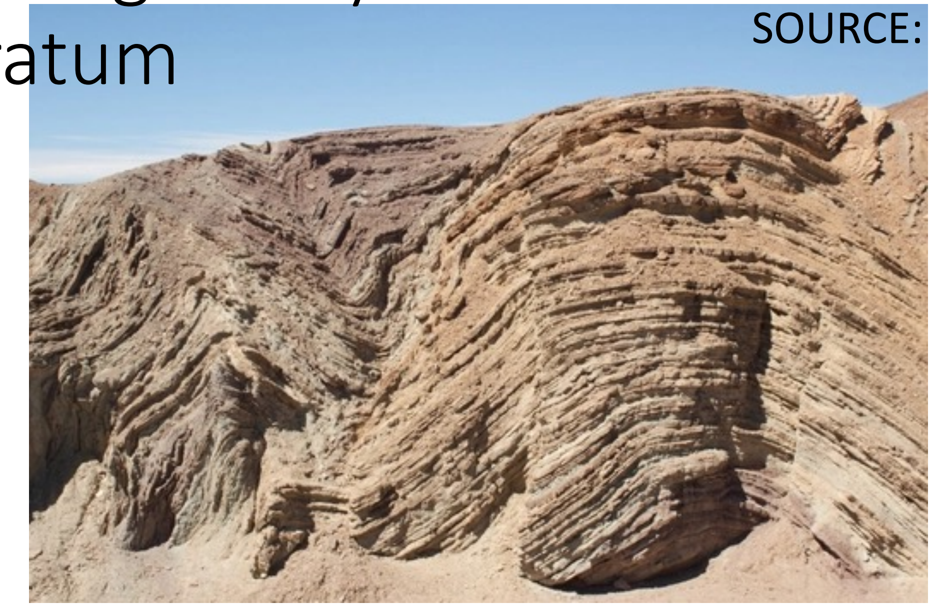
State of the art

Constrained elastica/beam/rod/plate/sheet……

Drilling industry



Geological layers: Stratum



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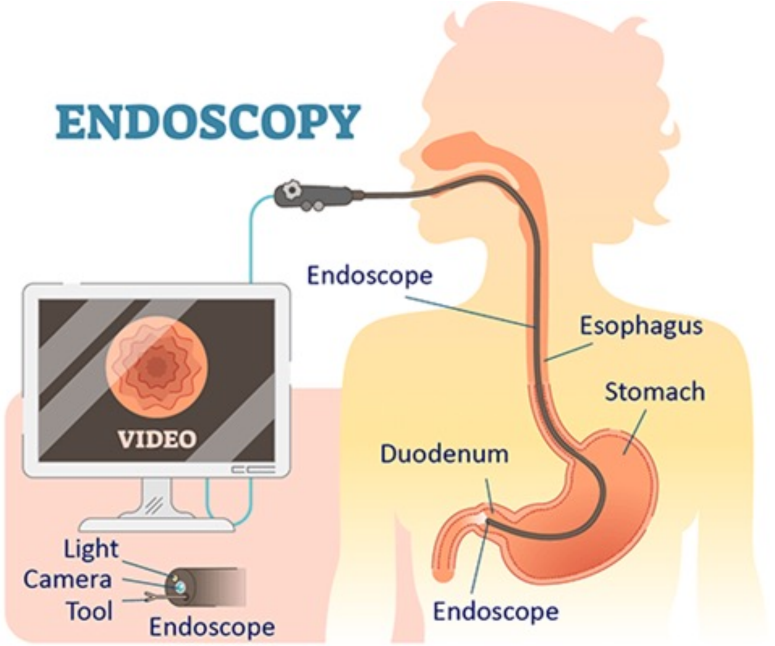

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J. Nonlinear Sci. Vol. 7: pp. 281–314 (1997)

Constrained Euler

ELSEVIER

Large defor
movable ar

Pengcheng Jia

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[b] Department of Mechan

Received

European Journal of Mechanics / A Solids 77 (2019) 103791

European Journal of Mechanics A/Solids

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ringy walls

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oi:10.1098/rspa.2005.1458
lished online 23 June 2005

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Yin-Yang spiraling transition of a confined buckled elastic sheet

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(Dated: December 15, 2023)

DNA in viral capsids, plant leaves in buds and geological folds are examples in nature of tightly packed low-dimensional objects. However, the general equations describing their deformations and stresses are challenging. We report experimental and theoretical results of a model configuration of compression of a confined elastic sheet, which can be conceptualized as a 1D line inside a 2D rectangular box. In this configuration, the two opposite ends of a planar sheet are pushed closer, while being confined in the orthogonal direction by two rigid walls separated by a given gap. Similar compaction of sheets has been previously studied, and was shown to buckle into quasi-periodic motifs. In our experiments, we observed a new phenomenon, namely the spontaneous instability of the sheet, leading to localization into a single Yin-Yang pattern. The linearized Euler Elastica theory of elastic rods, together with global energy considerations, allow us to predict the symmetry-breaking of the sheet in terms of the number of motifs, compression distance and tangential force. Surprisingly, the appearance of the Yin-Yang pattern does not require friction.

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which provide valuable insights and intuition. For example, we show that important features of the behavior, such as the transition from point contact to line contact and switching to the next mode, are dictated solely by a non-dimensional force, regardless of all other parameters of the system, and that the full description of the behavior is possible by means of two non-dimensional quantities that describe the relative stiffness of the nonlinear spring compared to that of the beam. The results also highlight the fundamental differences between the behavior with a stiffening spring or with a softening spring, such as the number of attainable modes and the monotonicity of the overall force–displacement relation. These results are then validated by experiments. [DOI: 10.1115/1.4064684]

Keywords: post-buckling; contact, constrained elastica, mathematical modeling, experiments, structures

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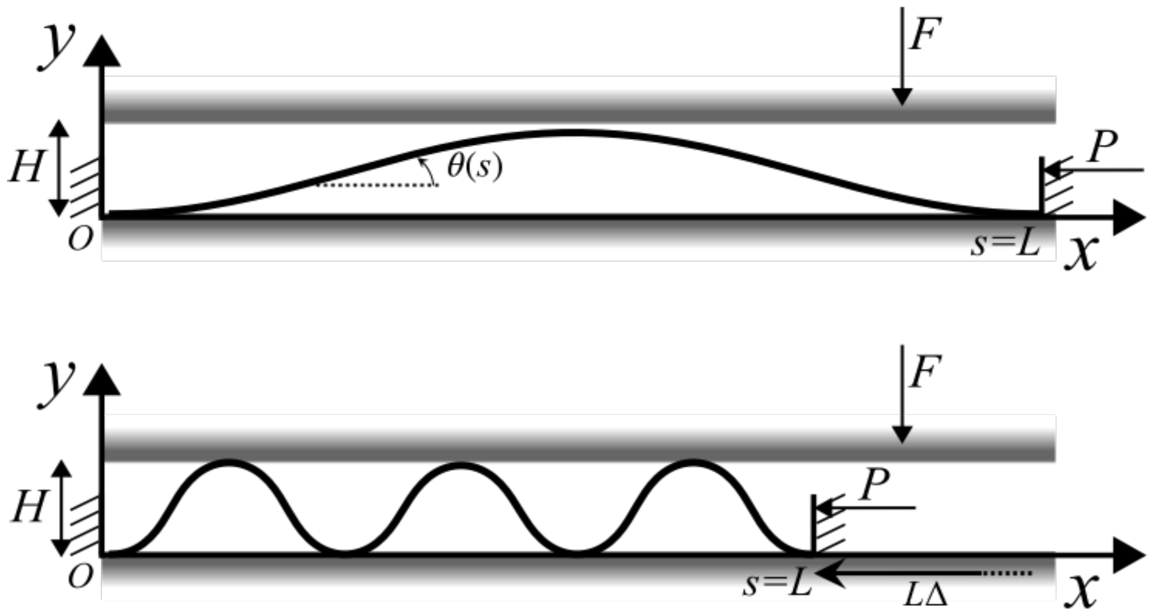
A sheet confined between rigid walls

- Length: L, width: w, thickness: t
- Bending Stiffness: $D = \frac{1}{12}Et^3$
- Boundary: clamped
- Displacement control: $L\Delta$
- Wall height: H
- Force response:
  - P (horizontal), F (vertical)
- Geometry:
  - Arc length: s
  - Curvature: $\kappa(s)$
  - Deflection angle: $\theta(s)$
  - Positions: $x(s)$, $y(s)$



Non-dimensionalization:
- Unit length: L
- Unit force: D/L

In the following presentation, all variables are non-dimensional

Numerical implementation & Results

Modelling as an optimization problem:

$$\min_{\kappa,\theta,x,y} \mathcal{E}_{tot} = \int_0^L \frac{\theta'}{2} ds$$

With constraints:

- Geometry

$$x'(s) = \cos\theta(s),$$
$$y'(s) = \sin\theta(s),$$

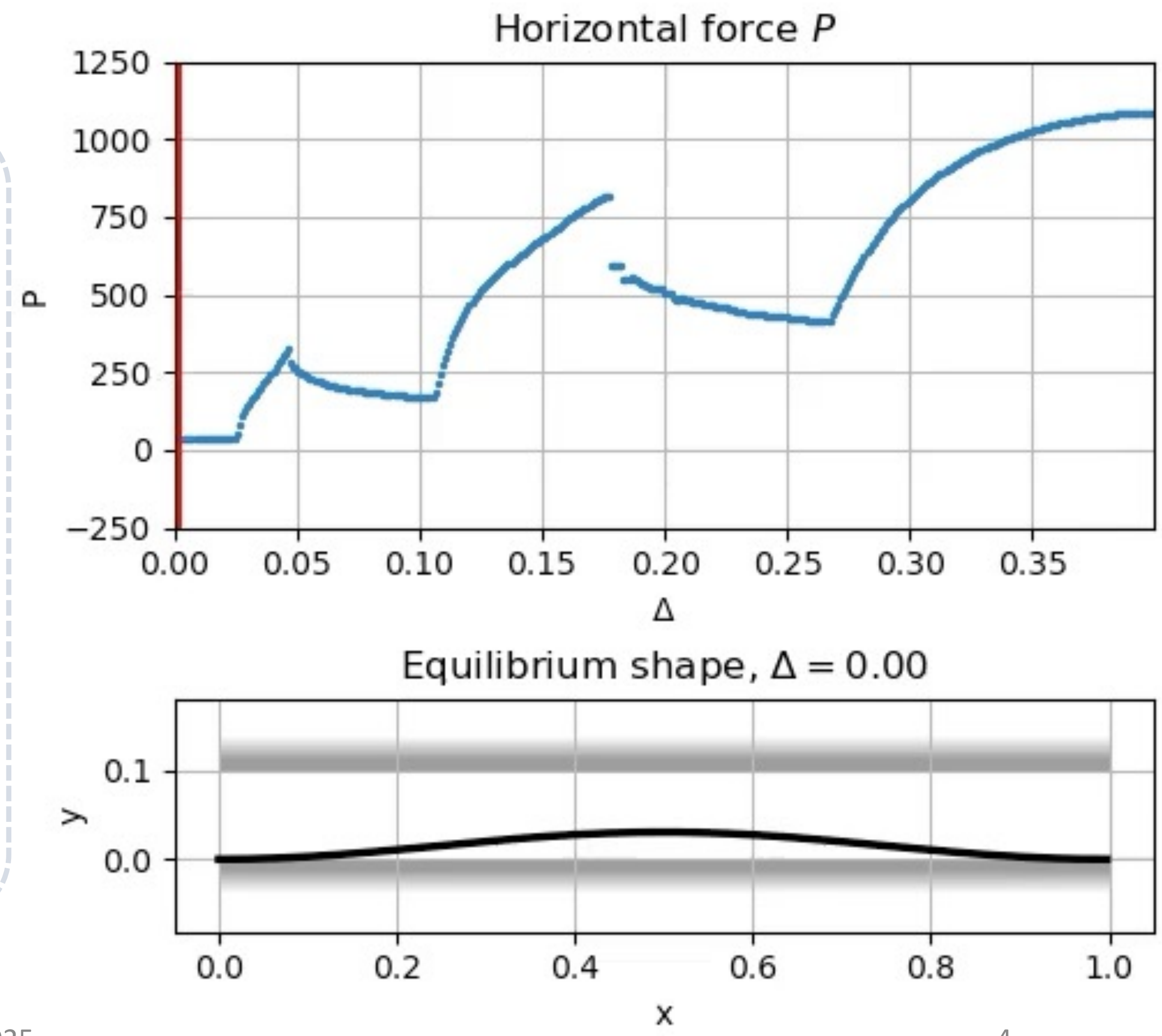
- Wall constraints

$$0 \le y(s) \le H, \qquad \forall s \in [0,1]$$

- Boundary conditions

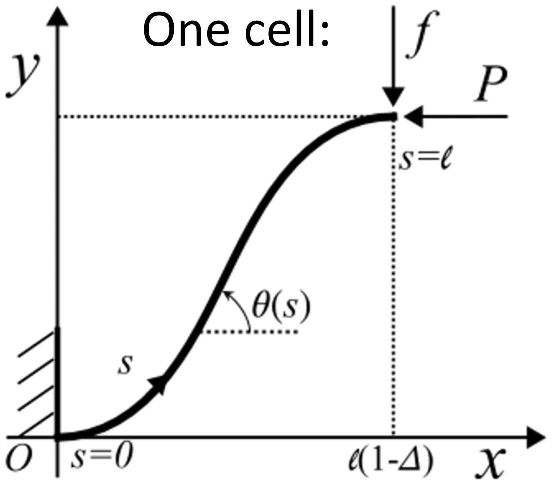
$$\theta(0) = 0, x(0) = y(0) = 0$$
$$\theta(1) = 0, y(1) = 0, x(1) = 1 - \Delta$$

The problem is solved as an optimization problem, discretized and implemented in Python, using IPOPT (**I**nterior **P**oint **O**ptimizer) with CasADi



A general solution describing the behaviors: the cellular model

n ×



Constructing the Lagrangian :

$$\mathcal{L}(\theta(s), x(s), y(s), \ell) = \underbrace{\frac{L}{\ell}}_{n}\left\{\int_{s=0}^{\ell} \frac{\theta'^2}{2} - [P(x' - \cos\theta) + f(y' - \sin\theta)]\, ds\right\}$$

$$\frac{\delta\mathcal{L}}{\delta(\cdot)} \Rightarrow$$

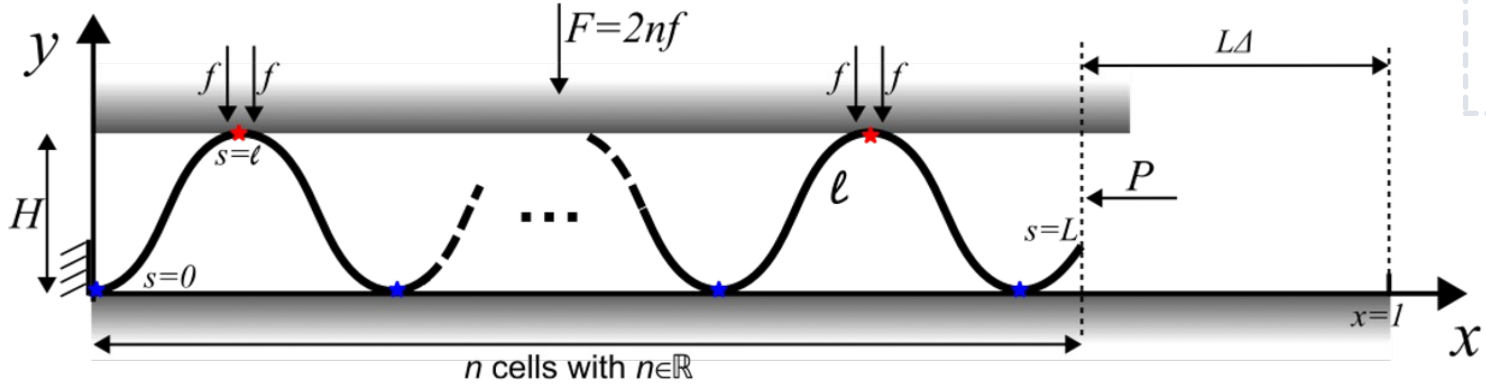
Differential equation system :

$$x'(s) = \cos\theta\,(s),$$

$$y'(s) = \sin\theta\,(s),$$

$$\theta''(s) = -P\sin\theta(s) + f\cos\theta(s)$$

$$\frac{\partial\mathcal{L}}{\partial l} \Rightarrow \boxed{\frac{1}{2}\theta'^2(\ell) = P\cdot\Delta - \frac{f}{2}\frac{y(\ell)}{\ell}}$$



With boundary conditions

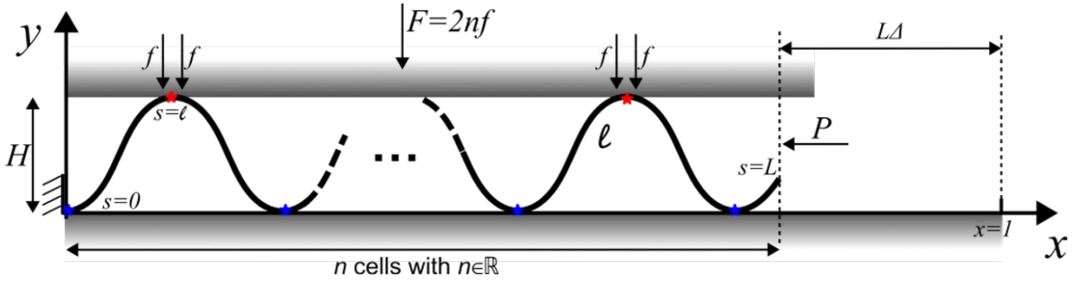
$$\theta(0) = 0, x(0) = y(0) = 0$$

$$\theta(\ell) = 0, y(\ell) = H, x(\ell) = \ell(1-\Delta)$$

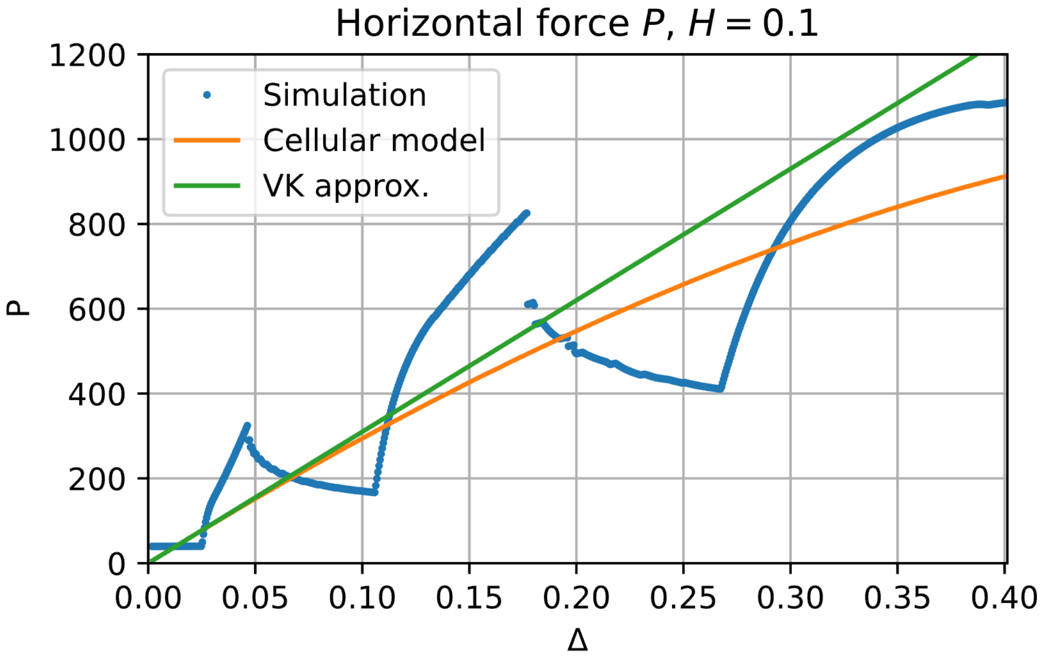
Cellular model with Von Karman approximation

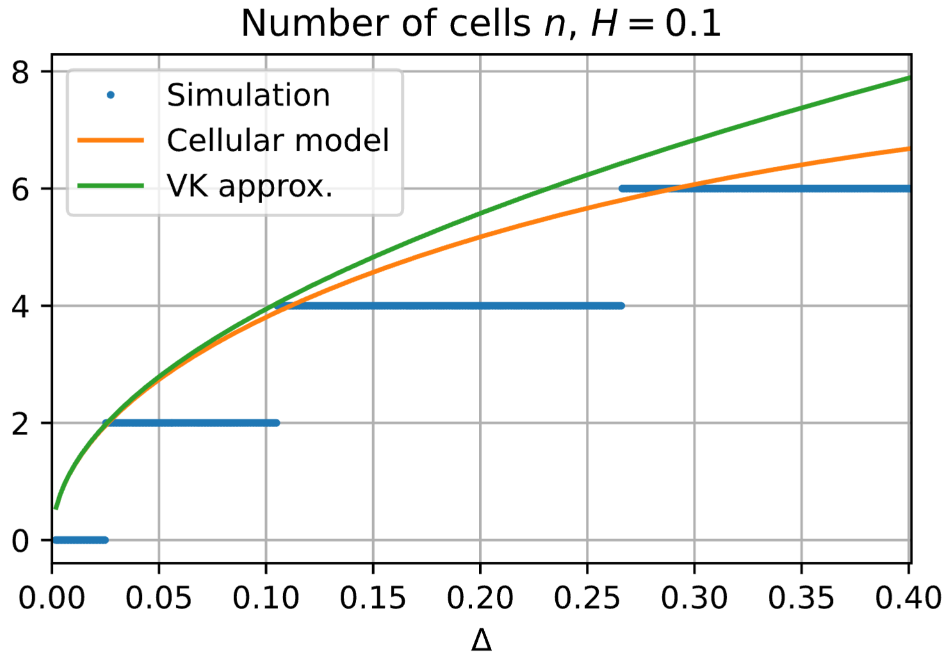
- Von Karman approximation:

$$x'(s) = \cos\theta(s) \approx 1 - \frac{\theta^2(s)}{2}$$
$$y'(s) = \sin\theta(s) \approx \theta(s)$$



- Force responses: $P = 31\frac{\Delta}{H^2}$
- Number of cells: $n = 1.247\frac{\sqrt{\Delta}}{H}$





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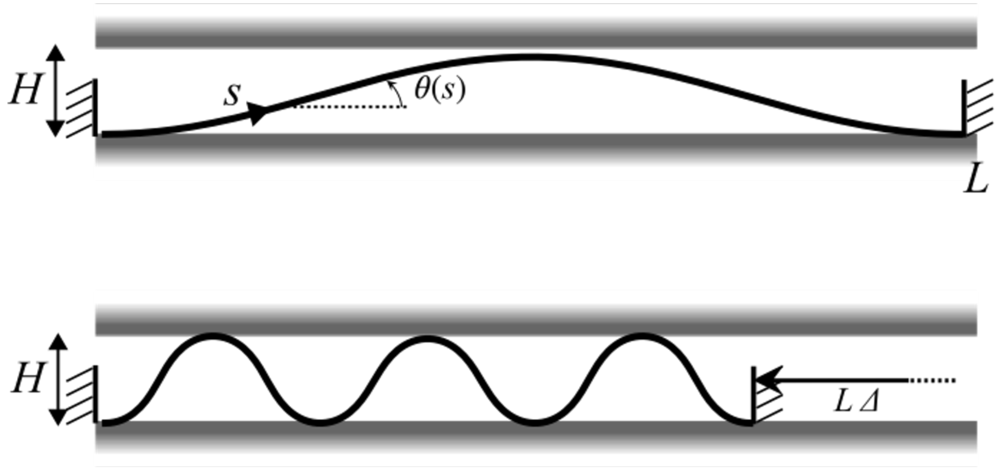
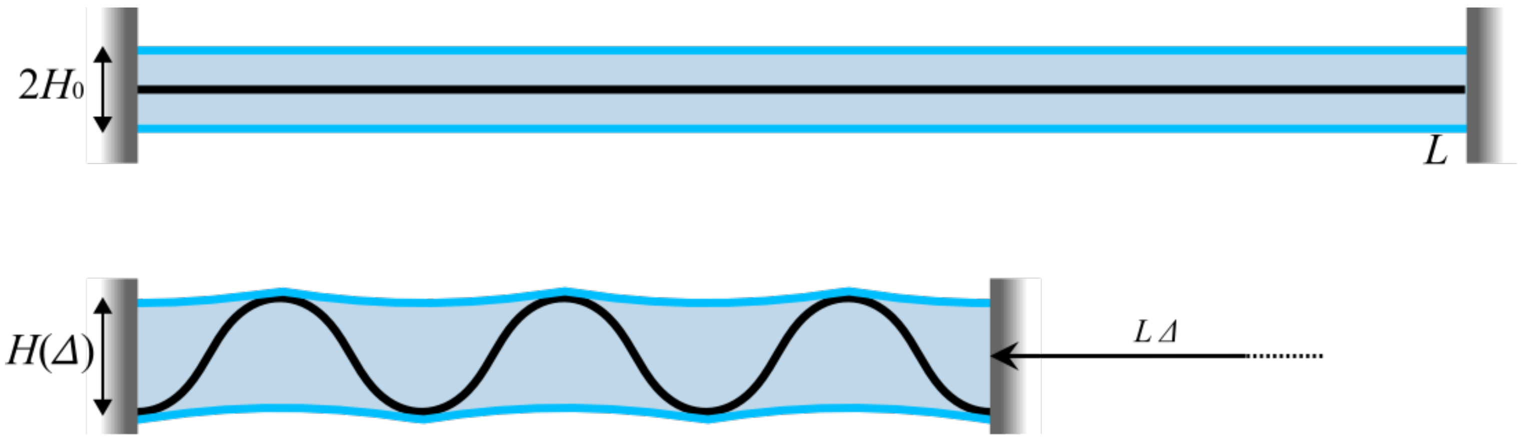
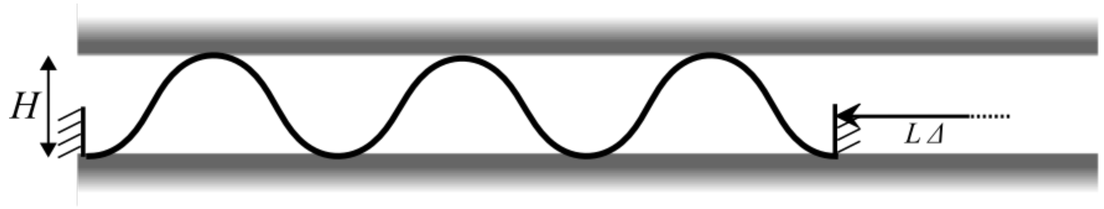


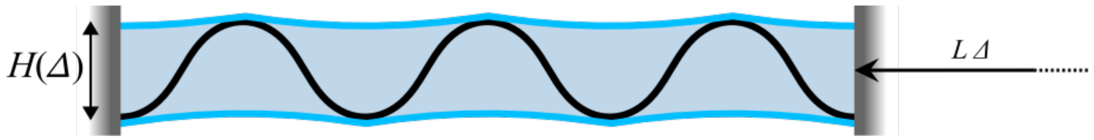
Rigid walls

How about "liquid walls"?

Liquid walls







Confined between rigid walls

Objective:

$$\min \mathcal{E}_{\text{elastic}}$$

With constraints:

- Geometry + Boundary conditions
- Wall constraints
$$0 \leq y(s) \leq H$$

Confined between liquid walls

Objective:

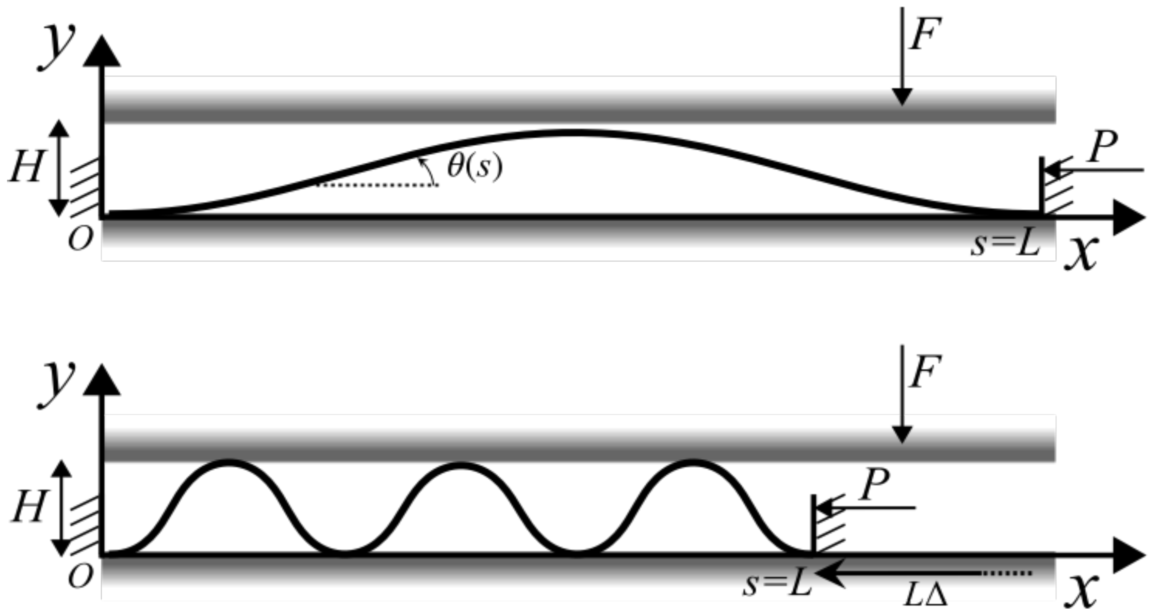
$$\min \mathcal{E}_{\text{elastic}} + \mathcal{E}_{\text{surface}}$$

With constraints:

- Geometry + Boundary conditions
- No penetration constraints
$$y_{\text{lower}}(x) \leq y(x) \leq y_{\text{upper}}(x)$$
- Constant volume

A sheet confined between rigid walls

- Length: L, width: w, thickness: t
- Bending Stiffness: $D = \frac{1}{12}Et^3$
- Boundary: clamped
- Displacement control: $L\Delta$
- Wall height: H
- Force response:
    - P (horizontal), F (vertical)
- Geometry:
    - Arc length: s
    - Curvature: $\kappa(s)$
    - Deflection angle: $\theta(s)$
    - Positions: $x(s)$, $y(s)$

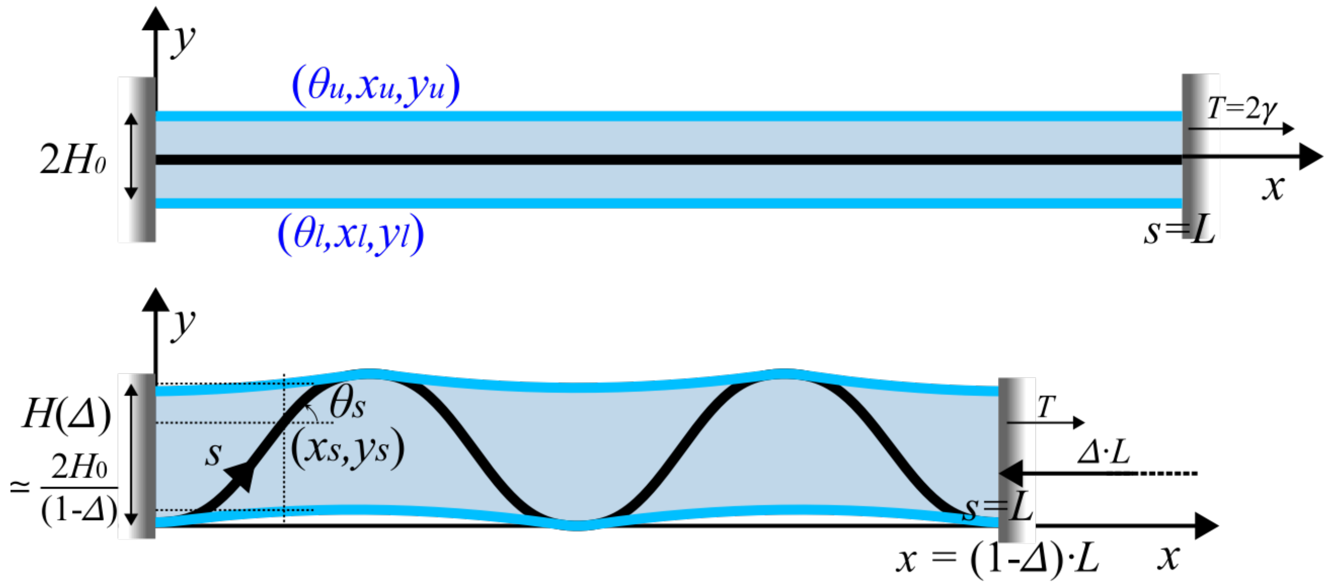


Non-dimensionalization:
- Unit length: L
- Unit force: D/L

In the following presentation, all variables are non-dimensional

A sheet confined between liquid walls

- Length: L, width: w, thickness: t
- Bending Stiffness: $D = \frac{1}{12}Et^3$
- Boundary: clamped, periodic
- Displacement control: $L\Delta$
- Fixed volume: $2LH_0$
- Surface tension: γ
- Force response: $T = 2\gamma - P$
- Geometry:
  - Elastic sheet: $(\kappa_s, \theta_s, x_s, y_s)$
  - Liquid interfaces:
    (θ_u, x_u, y_u)
    (θ_l, x_l, y_l)



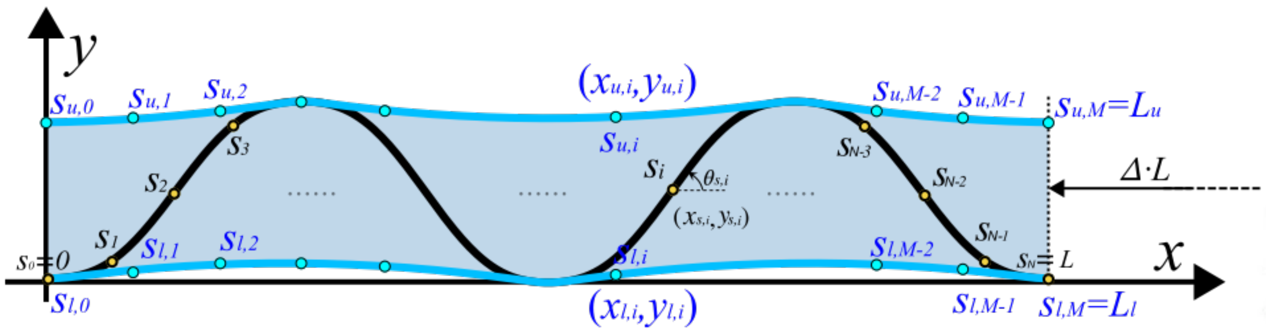
Non-dimensionalization:

- Unit length: $\sqrt{D/\gamma}$
  (Elasto-capillary length)
- Unit force: $\sqrt{D\gamma}$

<u>Dimensionless parameters:</u>

- Magnitude of capillary effects: $H_\gamma = \sqrt{\frac{\gamma}{D}}H_0$
- Membrane length vs. liquid volume: L/H_0
- Compression Δ

Numerical implementation and results



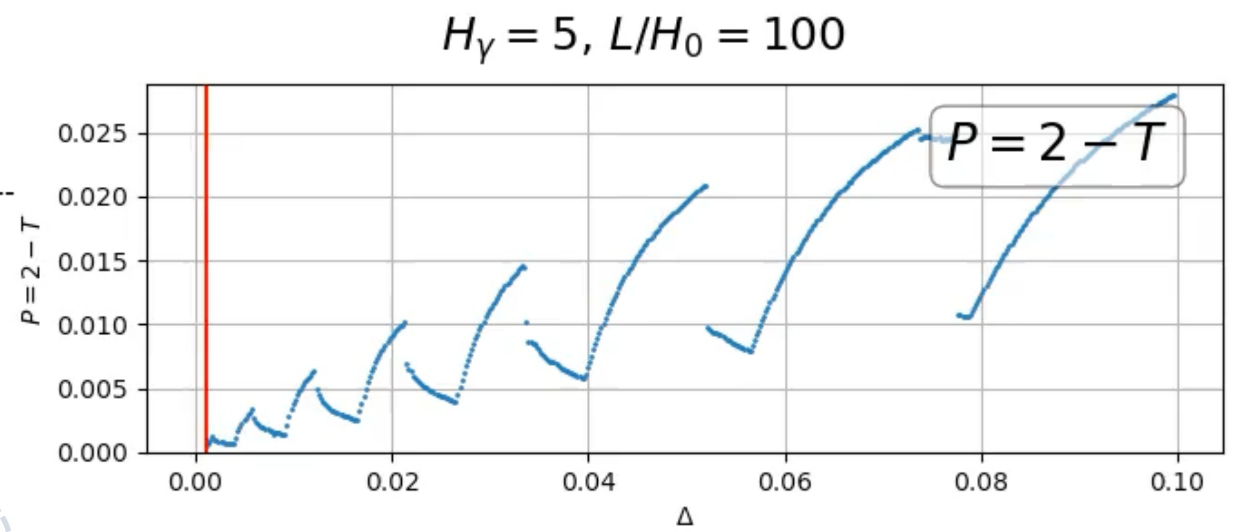
Objective:

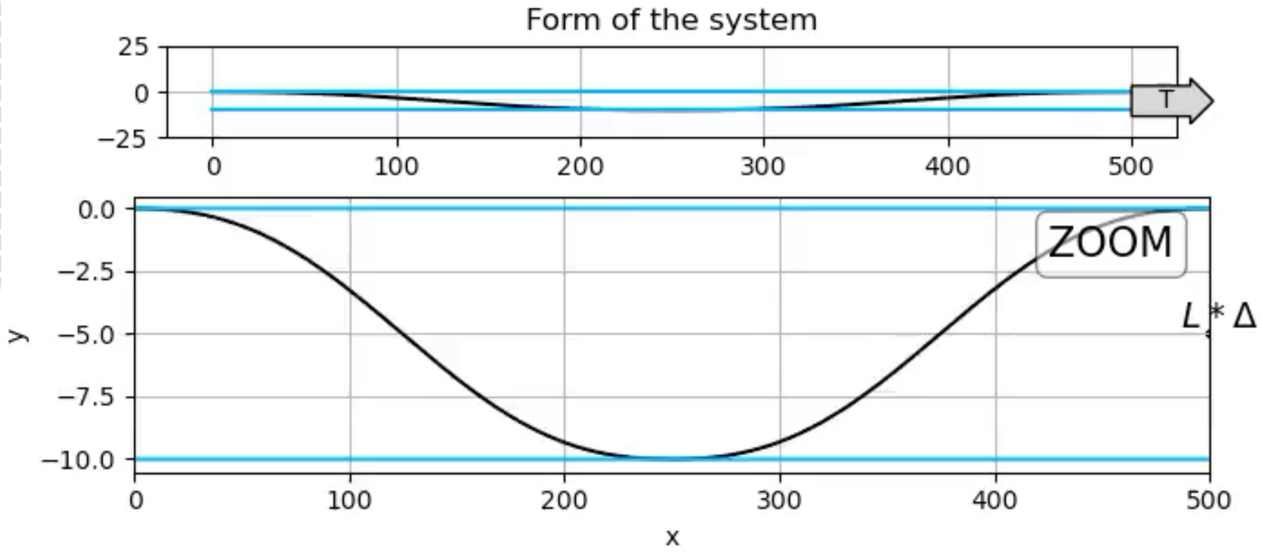
$$\min \mathcal{E}_{\text{elastic}} + \mathcal{E}_{\text{surface}}$$

With constraints:

- Geometry + Boundary conditions
- No penetration constraints
  $$y_{\text{lower}}(x) \leq y(x) \leq y_{\text{upper}}(x)$$
- Constant volume

Implemented in Python, using IPOPT and **CasADi**





Back to the cellular model: self-similarity

<u>Original cellular model:</u>

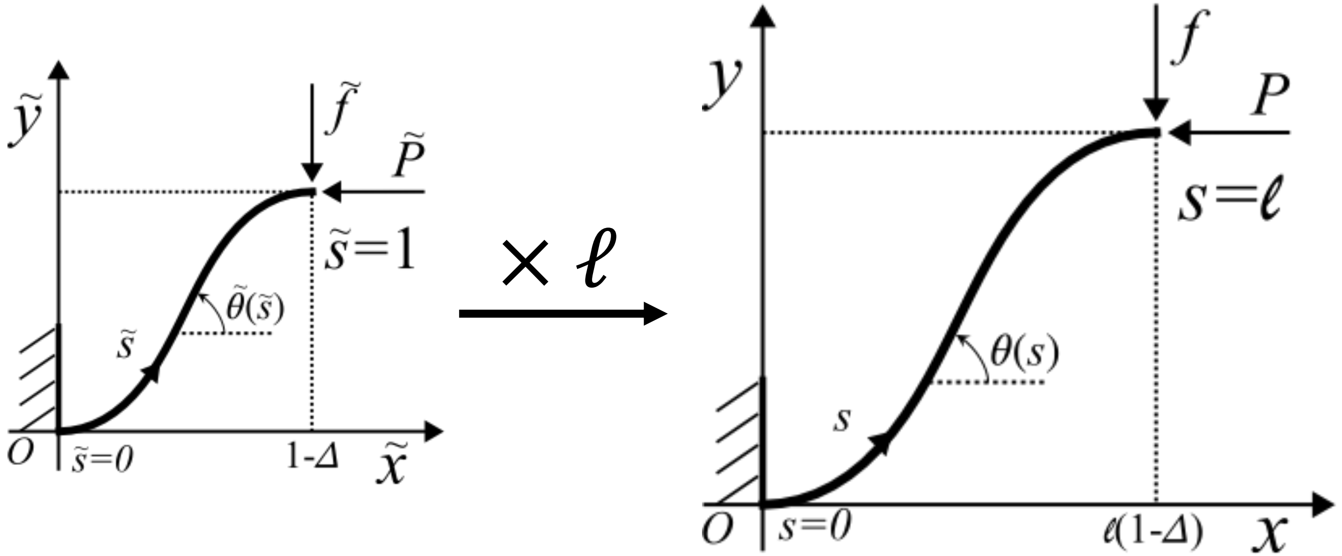
Differential equation system with boundary conditions:

$$\theta(0) = \theta(\ell) = 0,$$
$$x(0) = y(0) = 0$$
$$x(\ell) = \ell(1 - \Delta), \qquad \boxed{y(\ell) = H}$$

<u>Self-similar system:</u>

Differential equation system with boundary conditions:

$$\tilde{\theta}(0) = \tilde{\theta}(1) = 0,$$
$$\tilde{x}(0) = \tilde{y}(0) = 0$$
$$\tilde{x}(1) = 1 - \Delta$$



One universal solution!

How to find the scale ℓ?

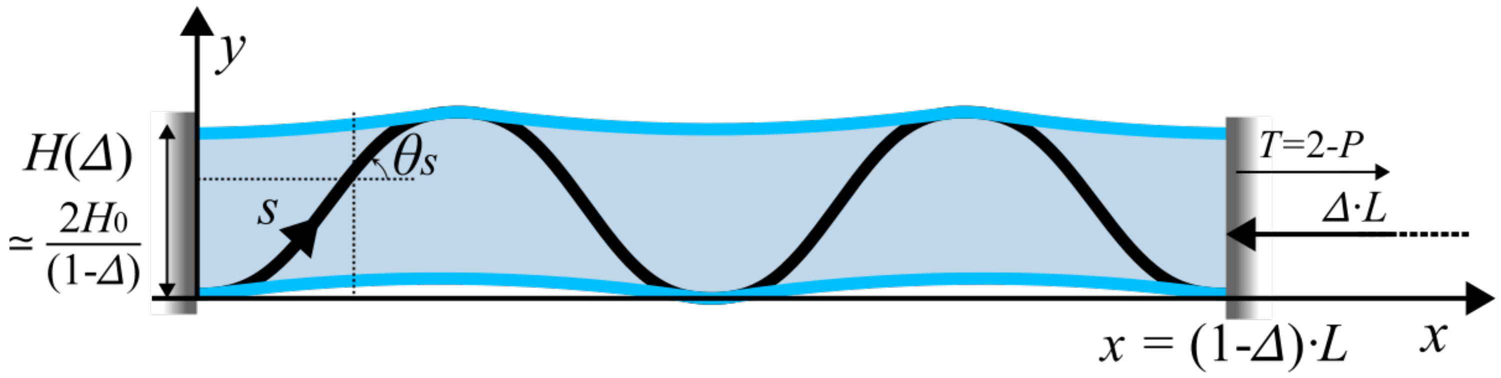
- Solid walls:

$$y(\ell) = \ell\tilde{y}(1) = H$$

- Liquid walls:
- From volume constraint

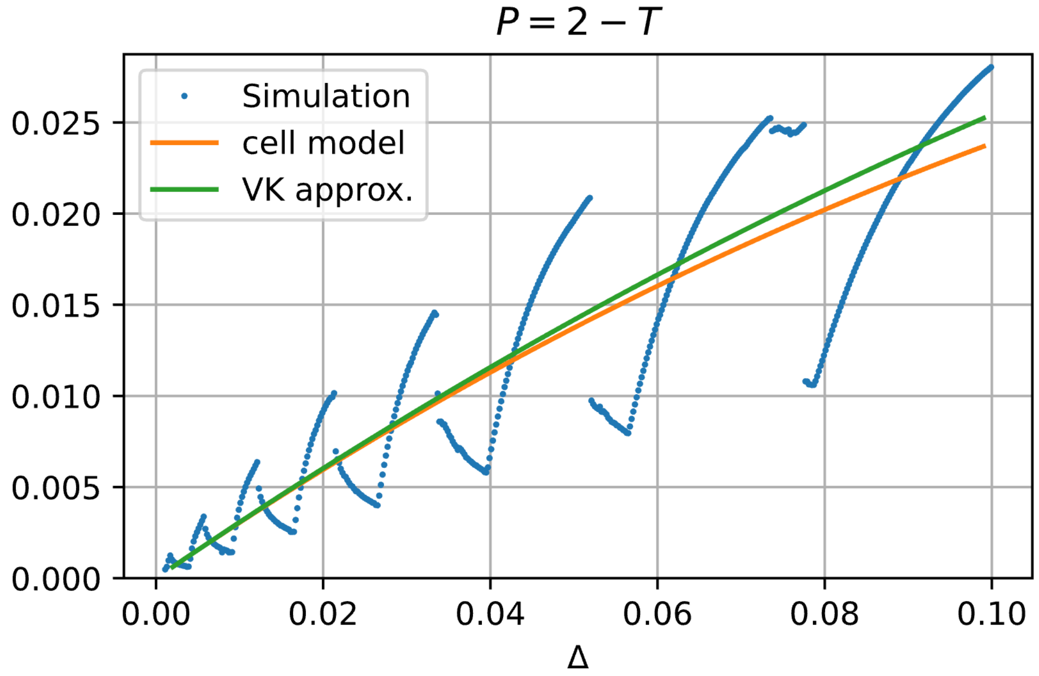
$$x(\ell)y(\ell) = 2H_0\ell$$
$$\Longrightarrow y(\ell) = \ell\tilde{y}(1) = \frac{2\,H_0}{1-\Delta}$$

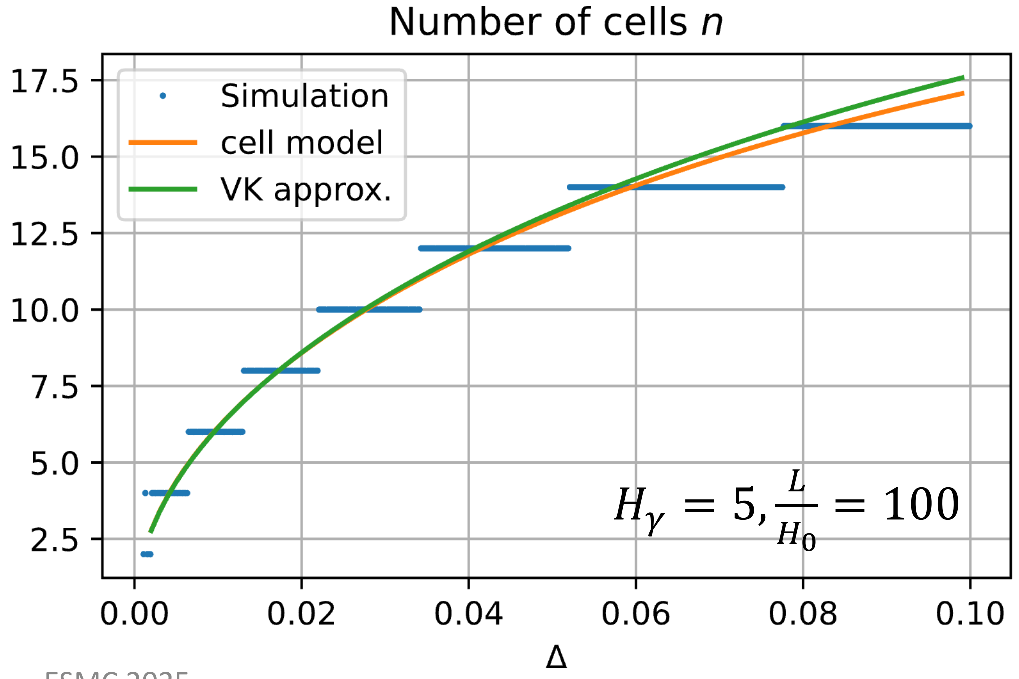
Self-similarity also applies to the Von Karman approximation



For the liquid walls:

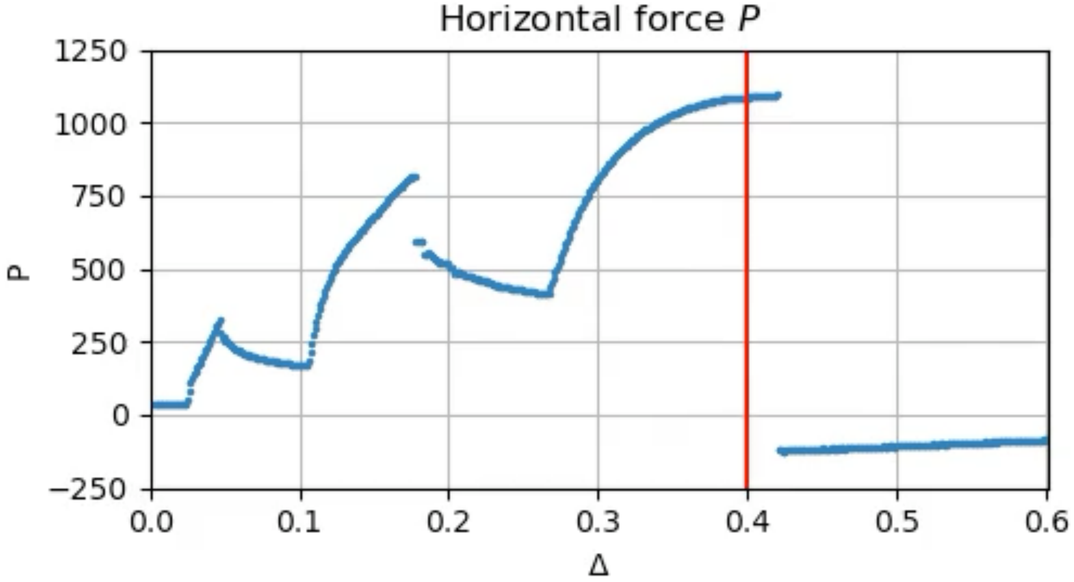
- Force responses: $P = 7.85 \frac{\Delta(1-\Delta)^2}{H_\gamma^2}$
- Number of cells: $n = 0.62\sqrt{\Delta}(1-\Delta)\frac{L}{H_0}$

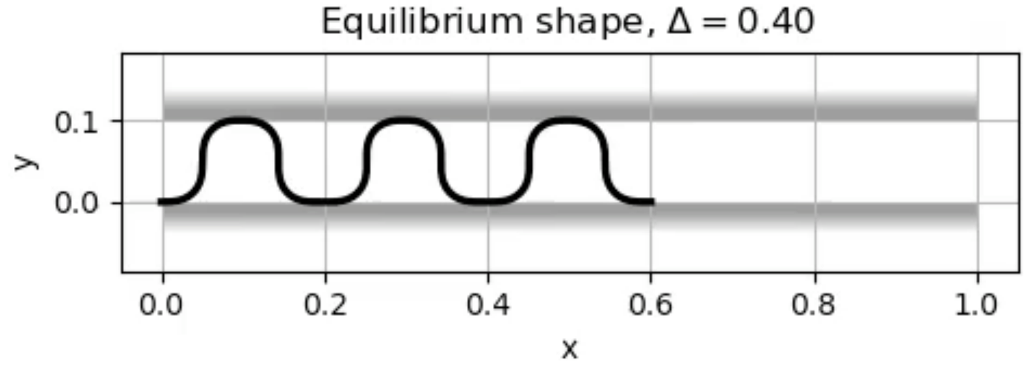




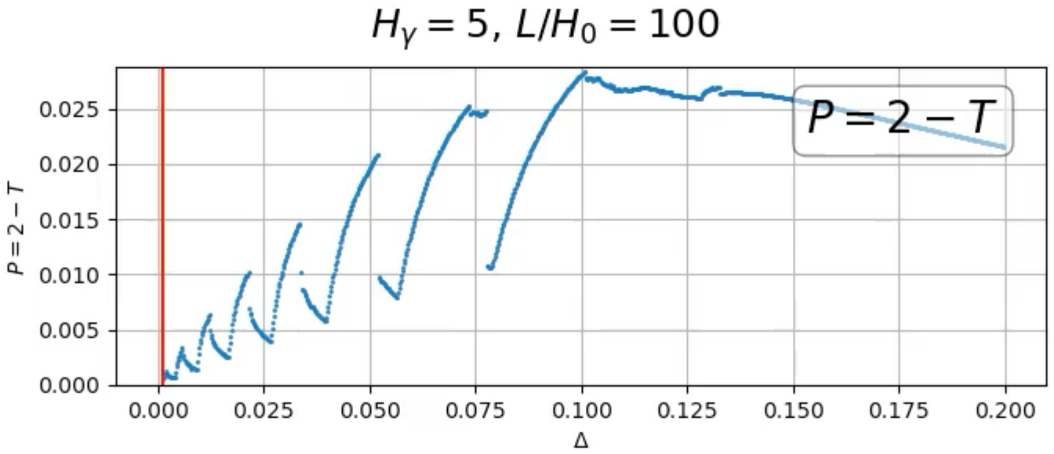
Beyond the Periodic Folding regime

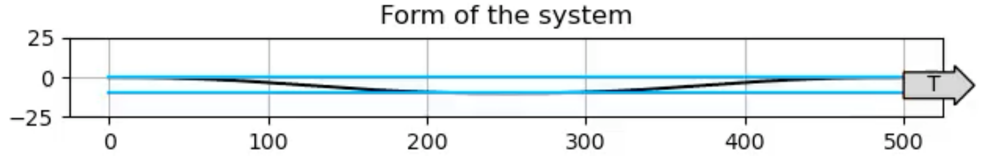
Confined between solid walls

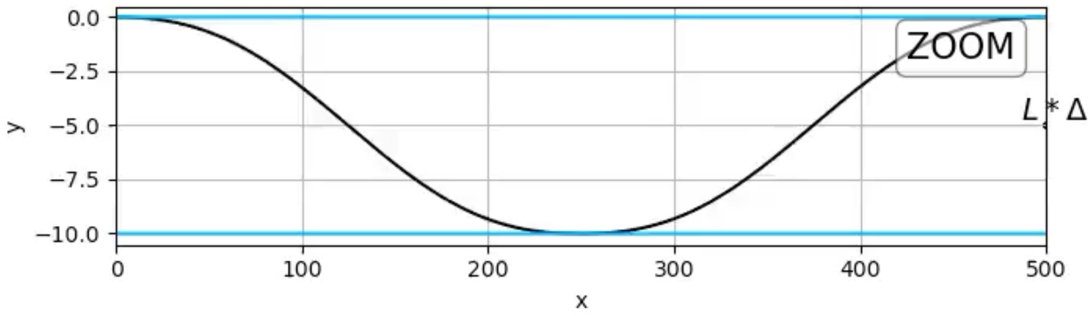




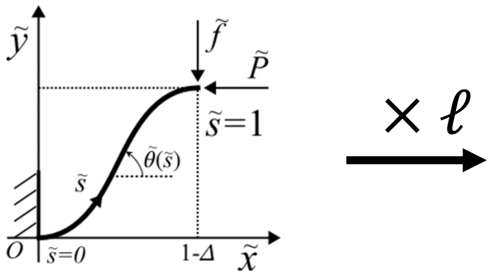
Confined between liquid walls

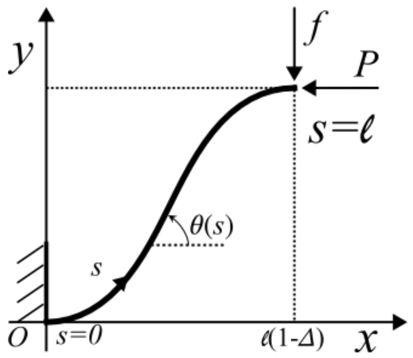






Explanation by cellular model





With $\ell = \dfrac{2\,H_0}{(1-\Delta)\tilde{y}(1)}$

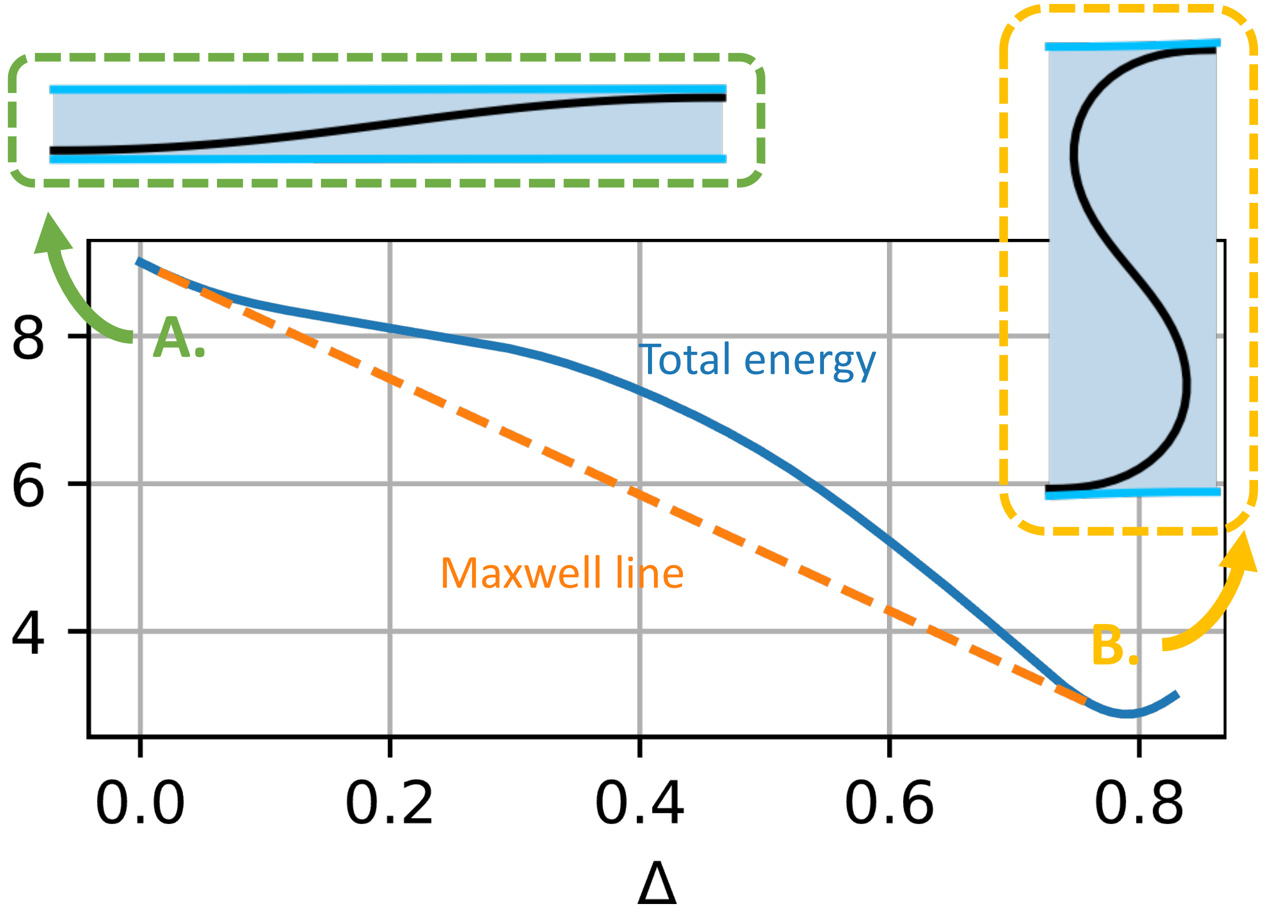
- We also have the universal solution for the elastic energy:

$$\tilde{\mathcal{E}}_{\text{cell}} = \int_0^1 \frac{1}{2}\,\hat{\theta}'^2(\tilde{s})\,d\tilde{s}$$

And the total energy is

$$\mathcal{E}_{\text{tot}} = \mathcal{E}_{\text{elastic}} + \mathcal{E}_{\text{surface}}$$

$$= \underbrace{\frac{L}{\ell}}_{n}\underbrace{\frac{\tilde{\mathcal{E}}_{\text{cell}}}{\ell}}_{\mathcal{E}_{\text{cell}}} + 2L(1-\Delta)$$



Thank you for your attention!

PhD defense: Oct. 2025 ⟶ YouTube @DAlembert-SU-CNRS